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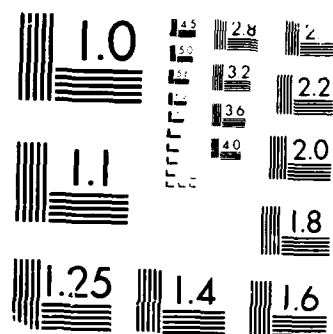
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## Final Technical Report

AFOSR Grant 84-0285

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The physical phenomena described by nonlinear partial differential equations have become at present the central theme of investigations by many researchers. A good understanding of most physical processes requires accounting for nonlinear effects and, consequently, methods for studying nonlinear equations have to be developed.

Among nonlinear equations the Dirichlet problem for the Monge-Ampere equation is the model case for fully nonlinear equations. The problem is formulated as follows.

In Euclidean plane  $R^2$  with Cartesian coordinates  $x, y$  consider a bounded domain  $\Omega$ , a nonnegative function  $f: \Omega \rightarrow [0, \infty)$ , and a continuous function  $\phi: \partial\Omega \rightarrow R$ . It is required to investigate solubility of the problem

$$M(z) = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = f \text{ in } \Omega, \quad (1)$$

$$z|_{\Gamma} = \phi, \Gamma = \partial\Omega. \quad (2)$$

The equation (1) is perhaps the most simple representative of the class of nonlinear equations of Monge-Ampere type. Such equations have been studied by many authors, mainly in connection with problems of existence and uniqueness of surfaces with prescribed metric or curvature functions. However, they also have other important applications. In particular, the leading term in the "balance equation" in dynamic meteorology has the form (1). In a more complicated form, an equation of this type appears in the von Karman system of equations for elasticity and also in an inverse problem of geometric optics; see below part B. It also turned out that recent progress in the study of fully nonlinear equations became possible after important properties of the equation (1) were discovered. For quasilinear elliptic and parabolic equations the use of equations of the form (1) is crucial in obtaining  $C^0$  and Hölder estimates.

In spite of the increasing number of papers in this area the theory of the problem (1), (2) and its generalizations is far from being complete. Researchers in the USA, USSR, Germany, England and other countries at present are actively pursuing this direction.

The proposers were fortunate to start their research at a relatively early stage of all these developments. The project funded by the AFOSR Grant 84-0285 covers two major areas:

- A. Investigation of numerical methods for solving problem (1), (2) and its generalizations.
- B. Investigation of solubility of a Monge-Ampere equation arising in shaped antenna design.

**Part A. Numerical Methods.** The type of the operator  $M$  depends on the function on which it is evaluated. For that reason, one usually seeks a solution of (1), (2) in the class of functions on which  $M$  has a fixed type, for example, elliptic or hyperbolic. In the case under consideration, the requirement  $f > 0$  in  $\Omega$  forces any function satisfying (1) to be an "elliptic" solution, that is, in the class where  $M$  is elliptic.

In view of the important practical applications several heuristic approaches were suggested for numerical solution of some modified forms of (1), (2). Though no rigorous analysis of these methods exists, one may note that they all are local methods based on a finite difference approximation and linearization. Because of the strong nonlinearity of  $M$  this approach might be successful only in a neighborhood of the true solution and therefore, if a priori a good initial approximation is not available, these methods will not produce, generally, a sequence converging to the true solution.

We investigated this problem in detail and obtained the following results:

- A special discretization scheme for (1), (2) was suggested different from standard finite element or finite difference schemes. It can be shown that latter ones in known forms will not work here;
- an iterative method has been developed for solving the discretized version of (1), (2);
- the question of finding an initial approximation in our scheme is completely and effectively resolved; it is just a routine step of the iteration process;
- the iterations are selfcorrecting.
- global convergence is established.

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- our algorithm is suitable for a parallel computer;
- a computer code has been written and tested on a serial machine.

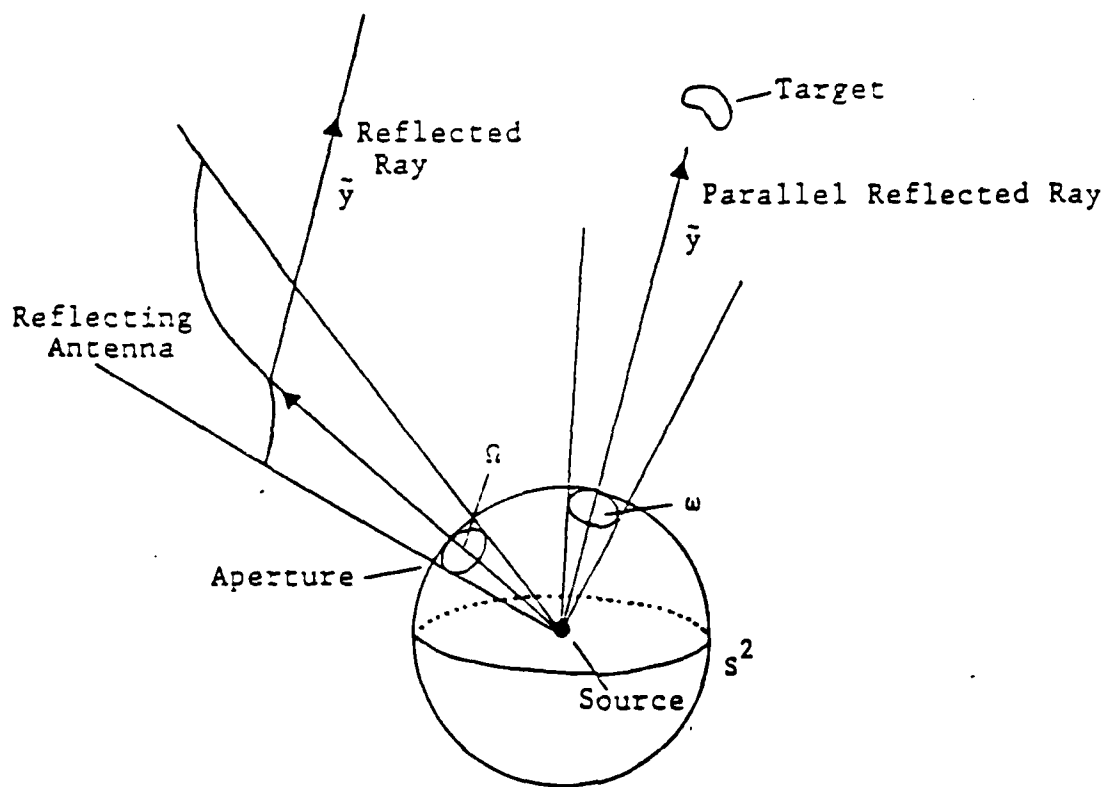
The experience gained in testing our procedure on different types of examples, including ones with large gradients, shows that its most effective use will be in combination with some fast Newton-type scheme. More precisely, a particular criteria exists for checking when the current approximation can be used as the beginning step for a converging Newton-type iterative procedure. In this combined scheme we proved convergence (quadratic) of the Newton iterates, but the above mentioned criteria involves some heuristic arguments and more work needs to be done here.

The computer code for the method is quite sophisticated; it involves, as a step, construction of a convex hull of sets of points in  $R^3$ . There are here different approaches and the effectiveness of the algorithm depends on it. We have been currently testing various schemes and the code presently is a substantial improvement over its original version of 1984-85.

The results have been submitted for publication (see the subsection on publications and presentations).

**Part B. Applications to Shaped Antenna Design.** In a practical problem of shaped antenna design it is required to determine a reflecting surface such that for a given point-source of light the reflected rays cover a prescribed region of the far sphere and the density of the distribution of reflected rays is a prescribed in advance function of the direction. It is assumed that the power density of the source as well as the aperture of the incident ray cone are known, and the reflection process obeys the laws of geometric optics (see Figure 1 on next page). In this form the problem was posed by Westcott and Norris and later it has been considered by Brickell, Marder, and Westcott. The research of these investigators has been supported by Plessey Radar, Ltd. for many years.

The problem admits a precise mathematical formulation and in this form it reduces to solving the equation



OBJECT: To Illuminate Target with Prescribed Intensity  
(Through Given Aperture)

Equivalent to Determining the Surface and Position of Reflector

Figure 1

$$\frac{4P^2 \det[\nabla_{ij}P + (P - \frac{|\nabla P|^2 + P^2}{2P})e_{ij}]}{(|\nabla P|^2 + P^2)^2 \det(e_{ij})} = f \text{ in } \omega, \quad (3)$$

$$\text{nonlinear boundary condition on } \partial\omega, \quad (4)$$

with respect to the unknown function  $P (>0)$  naturally associated with the problem; here  $(e_{ij})$  is the matrix of the first fundamental form  $e$  of the unit sphere  $S^2$ ,  $\nabla$  the gradient in the metric  $e$ ,  $\nabla_{ij}$  - second covariant derivatives in  $e$ , and  $f$  the prescribed power density. The condition (4) is somewhat complicated to be presented here without considerable expansion. The question of existence of solutions to this problem is open. Conditions for uniqueness were given by Marder. Westcott and Norris have given some results of numerical studies and applications. Still there are no rigorous convergence results, and even the linearized version of the problem has not been investigated. In 1957 J. Keller obtained results pertaining to the radially symmetric case without satisfying a particular boundary condition.

With the support of AFOSR we started our investigation of the problem with the radially symmetric (r.s.) case, that is, when the incidence ray cone  $\Omega$  and the far field domain  $\omega$  are circular, the prescribed density of reflected rays is a function of the azimuthal angle only, and the reflecting surface is sought as a surface of revolution, later we also investigated the nonradially symmetric case. The following results have been obtained:

- The problem splits naturally into two parts. In the first part one finds a class of surfaces for which the reflected directions cover the prescribed domain  $\bar{\omega}$ . Those surfaces can be conveniently parametrized by points of  $\bar{\omega}$ . In the second part one seeks in the above class a particular surface for which the density of the reflected rays is a prescribed function in  $\omega$ , and which projects onto the given domain  $\bar{\omega}$ ;
- it is shown that for any function  $p \in C^2(\bar{\omega})$ ,  $p > 0$  in  $\bar{\omega}$ , the surface  $F$  defined by the map

$$-r = \nabla p + (p - \rho)y, \quad y \in \bar{\omega}, \quad (5)$$

$$\rho = (p^2 - |\nabla p|^2)^{1/2}, \quad |\nabla p|^2 = \langle \nabla p, \nabla p \rangle,$$

satisfies the requirement of being a reflector:



- simple and verifiable necessary and sufficient conditions for solubility of (3), (4). It was shown that radially symmetric solutions can be constructed explicitly whenever all of the parameters are appropriate.

In a separate paper we proved that in the nonradially symmetric case with circular incident ray cone and far field and distribution density close (in certain norm) to a radially symmetric distribution this problem admits two classes of solutions provided the data satisfies a necessary condition expressing the energy conservation law.

The methods of this paper could be used to find solutions to the above problem in cases when the incident ray cone and the far field are not circular but close to such (in certain particular sense). Also, the main theorem here provides an important step in a continuation scheme for proving existence of solutions to the above problem with distribution densities not subject to the requirement of being close to the radially symmetric ones. We intend to return to these questions in subsequent publications. One other consequence of our main result here is that it justifies the use of Newton-type local methods for numerical solution of the problem.

During the past year we have also experimented with numerical methods for solving (3), (4) for  $\phi$  sufficiently close to a r.s. symmetric density. In r.s. case the equation is singular at the endpoint corresponding to axis of revolution, and as a result straight forward linearizations about r.s. solutions do not seem to work very well.

#### Publications and Presentation

The results obtained in Part A were presented in

- a paper "On the numerical solution of the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = f$$

and its discretizations, I". It has been submitted for publication and is currently under revision;

- a lecture at a special session of the Southeastern-Atlantic Regional Conference on Differential Equation, Atlanta, October, 1985

- a plenary lecture was delivered on the subject by one of the principal investigators at the South Eastern Regional SIAM Conference, March, 1986.

The results in Part B were presented in

- a paper "Radially symmetric solutions of a Monge-Ampere equation arising in a reflector mapping problem", pp. 1-19; accepted for publication in Proceedings of UAB Conference on Differential Equations and Mathematical Physics, Springer;
- a paper "Near radially symmetric solution of an inverse problem in geometric optics", pp. 1-23, to appear in the journal "Inverse problems", published by The Institute of Physics, England
- a report "On the Monge-Ampere equation arising in the reflector mapping problem", Institute for Mathematics and its Applications, University of Minnesota, preprint series #198, pp. 1-43 (this is an expanded version of the first paper mentioned in this part);
- an hour lecture given at the Technical University in Berlin, West Germany, December, 1985.
- a 30-minute presentation at the conference on Differential Equations and Mathematical Physics, University of Alabama, Birmingham, March, 1986.

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